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FORCES ON DISLOCATIONS AND INHOMOGENEITIES IN THE GAUGE THEORY OF DEFECTS

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Abstract—Previous studies have shown that the gauge theory of dislocations predicts that *effective* stress and *effective* linear momentum are the agencies that drive dislocation in finite bodies with applied tractions. The purpose of this article is to obtain an evaluation of the forces of interaction between the dislocations and the material properties for finite bodies with applied tractions. A careful examination of the variational arguments underlying such calculations is shown to provide explicit evaluations of the forces that arise because of spatial and temporal inhomogeneities. The interaction forces for finite material bodies are shown to have the same form as those for bodies of infinite spatial extent, provided the total stress and linear momentum are replaced by the effective stress and effective linear momentum. Implications of these results are discussed.

1. INTRODUCTION

The natural association of gauge transformations with the continuum theory of defects was first noted by Alicja Golebiewska-Lasota through an argument by analogy with electromagnetic theory[1]. Further investigations[2, 3, 4] revealed the intrinsic richness and order inherent in this Abelian gauge theory (gauge transformations of the first kind). The full theory was shown to admit a 45-fold Abelian gauge structure that could be used to obtain a unique decomposition into elastic and anelastic response modes. This structure also led to a natural four-dimensional space-time formulation of the continuum equations of dislocations and disclinations of extreme simplicity[4].

An argument by analogy, based on the four-dimensional formulation of the equations of defect dynamics, suggested a direct application of the Yang-Mills gauge theory construct to defect phenomena[4]. Extension of the Yang-Mills construct to the non-semisimple fundamental group $SO(3) \triangleright T(3)$ of elasticity theory, together with results stemming from a direct mapping of the equations of defect dynamics onto the equations of structure of E. and H. Cartan, gave rise to the gauge theory of defects[5, 6, 7]. The gauge theory with all of its many details is summarized in [8].

The basic theory of gauge fields leads to a direct description of material bodies of infinite spatial extent that support nontrivial dislocation and disclination fields. The transition to bodies of finite spatial extent with applied boundary tractions is still not fully understood if there are nontrivial disclination fields present. On the other hand, if the body is assumed to be disclination-free, a crafty use of degenerate (null) Lagrangians that accommodate inhomogeneous Neumann data (boundary tractions) can be used to obtain a complete and consistent formulation of traction boundary value problems with nontrivial dislocation fields [8 (Section 3.18), 9, 10]. This gives rise to a demonstration that effective stress and effective linear momentum, rather than total stress and total linear momentum, are the agencies that drive the dislocation fields.

The purpose of this study is to address the question of how to compute the forces that act on the dislocations in bodies of finite spatial extent with applied boundary tractions. As it turns out, the answer to this question also provides the means for calculating the forces that arise because of either spatial or temporial homogeneities of the body.

Sections 2 and 3 summarize the gauge theory of dislocations for finite material bodies subject to given initial and boundary data. The basis for computing forces of interaction and forces that arise from homogeneities is established in Section 4. The general results established in Section 4 are cut down to the specific problem of a finite body with dislocations in Section 5 (calculation of forces from inhomogeneities) and

Section 6 (forces of interaction). Interpretations and discussions of these forces and their relation to previous results are given.

2. NOTATION AND FIELD VARIABLES

Any material body under consideration is assumed to be referred to a natural reference configuration that is found in Euclidean three-dimensional space E_3 referred to Cartesian coordinates $\{X^A\} = \{X^1, X^2, X^3\}$. The natural volume element of E_3 is given by $\mu = dX^1 \wedge dX^2 \wedge dX^3$ and gives rise to the 2-forms of oriented surface $\{\mu_A\} = \{dX^2 \wedge dX^3, -dX^1 \wedge dX^3, dX^1 \wedge dX^2\}$. Here, $\wedge$ denotes the exterior product. The 4-dimensional space $E_3 \times \mathbf{R}$ with coordinates $\{X^a\} = \{X^A, T\} = \{X^1, X^2, X^3, X^4\}$ (i.e. $X^4 = T =$ time variable) is the natural space-time to which the dynamics of material bodies is referenced.

A dislocated body is characterized by the presence of nonzero dislocation densities and/or dislocation currents. These are fields of exterior forms that have the representations

$$\alpha^i = \alpha^{iA}(X^a)\,\mu_A = \text{dislocation density 2-forms}, \tag{2.1}$$

$$J^i = J^i_A(X^a)\,dX^A = \text{dislocation current 1-forms}. \tag{2.2}$$

They are determined in terms of

$$\beta^i = \beta^i_A(X^a)\,dX^A = \text{distortion 1-forms}, \tag{2.3}$$

$$V^i(X^a) = \text{distortion velocity 0-forms} \tag{2.4}$$

through the relations

$$\alpha^{Ai} = \epsilon^{ABC}\,\partial_B\beta^i_C, \qquad J^i_A = \partial_A V^i - \partial_4\beta^i_A. \tag{2.5}$$

The notation used here is defined by

$$\partial_A := \frac{\partial}{\partial X^A}, \qquad \partial_4 := \frac{\partial}{\partial X^4} = \frac{\partial}{\partial T},$$

and will be used throughout.

A marked convenience and an economy in notation is achieved through combining the above expressions by constructing certain exterior differential forms on the four-dimensional space-time $E_3 \times \mathbf{R}$. To this end, we combine (2.1) and (2.2) to give

$$D^i = J^i_A\,dX^A \wedge dT + \alpha^{Ai}\,\mu_A = \text{dislocation 2-forms}, \tag{2.6}$$

$$B^i = V^i\,dT + \beta^i_A\,dX^A \qquad = \text{velocity-distortion 1-forms}. \tag{2.7}$$

We may thus write

$$D^i = \frac{1}{2}\,D^i_{ab}\,dX^a \wedge dX^b, \qquad B^i = B^i_a\,dX^a, \tag{2.8}$$

in which case it is easily seen that the system (2.5) goes over into the particularly simple relations

$$D^i = dB^i, \tag{2.9}$$

where d denotes the process of exterior differentiation of differential forms on the four-dimensional space $E_3 \times \mathbf{R}$.

We have already remarked that the gauge theory of dislocations arises through use of the Yang-Mills construct that is demanded by breaking the homogeneity of the action of the translation group $T(3)$. The minimal replacement part of this construct replaces the displacement gradient 1-forms $d\chi^i = \partial_a\chi^i\, dX^a$ by the distortion 1-forms

$$B^i = d\chi^i + \phi^i, \tag{2.10}$$

where

$$\phi^i = \phi^i_a\, dX^a \tag{2.11}$$

are the Yang-Mills potential 1-forms that compensate for the inhomogeneous action of the translation group $T(3)$. When (2.11) are combined with the second of (2.8) and (2.7), this translates into

$$\beta^i_A = \partial_A\chi^i + \phi^i_A, \qquad V^i = \partial_4\chi^i + \phi^i_4. \tag{2.12}$$

It is then a simple matter to use (2.9) and (2.6) to obtain

$$\alpha^{Ai} = \epsilon^{ABC}(\partial_B\phi^i_C - \partial_C\phi^i_B), \qquad J^i_A = \partial_A\phi^i_4 - \partial_4\phi^i_A. \tag{2.13}$$

This shows that the state of dislocation of any material body is uniquely characterized by its compensating fields $\{\phi^i_a(X^b)\}$.

3. LAGRANGIAN FUNCTIONS AND THE FIELD EQUATIONS

The essential distinguishing feature of the gauge theory of dislocations is that it has an underlying variational principle in terms of the 15 field variables $\{\chi^i(X^b), \phi^i_a(X^b) \mid 1 \le i \le 3,\ 1 \le a \le 4\}$. The Lagrangian for this variational principle is not something that can be chosen in an arbitrary way, for it is uniquely determined by the elastic response properties of the body and the minimal replacement and minimal coupling constructs of the Yang-Mills procedure.

We start with the Lagrangian of an elastic body. For simplicity, let the strain energy of the body be quadratic in the strain measures

$$e_{AB} = (\partial_A\chi^i)\,\delta_{ij}\,(\partial_B\chi^j) - \delta_{AB} \tag{3.1}$$

and representative of an isotropic material. We then have

$$2l_0 = \rho_0(\partial_4\chi^i)\delta_{ij}(\partial_4\chi^j) - \frac{1}{4}\{\lambda(e_{AB}\delta^{AB})^2 + 2\mu e_{AB}\delta^{AC}\delta^{BF}\, e_{CF}\}, \tag{3.2}$$

where ρ_0 is the mass density in the reference configuration and λ, μ are the Lamé constants. When the reqired minimal replacement[8], $\partial_a\chi^i \mapsto \partial_a\chi^i + \phi^i_a = B^i_a$, is effected, we obtain the Lagrangian

$$2L_0 = \rho_0\, B^i_4\,\delta_{ij}\, B^j_4 - \frac{1}{4}\{\lambda(E_{AB}\delta^{AB})^2 + 2\mu E_{AB}\,\delta^{AC}\delta^{BF}\, E_{CF}\}. \tag{3.3}$$

where

$$E_{AB} = B^i_A\,\delta_{ij}\,B^j_B - \delta_{AB} \tag{3.4}$$

obtains from (3.1) under minimal replacement.

The Lagrangian L_0 will be referred to as the Lagrangian for a *free anelastic body*. The reader should carefully note that an L_0 can be obtained from any l_0 of an elastic body by this procedure. Thus, there is no difficulty in modeling anisotropic anelastic bodies and bodies with intrinsic nonlinear responses by this procedure. Simply change the stored energy contribution to l_0 in the appropriate way and then proceed with the minimal replacement $\partial_a\chi^i \mapsto \partial_a\chi^i + \phi^i_a$. The choice (3.2) has been made solely in the interests of simplicity, so that a definite class of material properties can be discussed. The reader can, however, replace the terms containing the Lamé constants in (3.2) by $\frac{1}{8}\, e_{AB}\, C^{ABCF}\, e_{CF}$ or any other strain energy function and the theory goes right through, albeit different constitutive relations.

The reason for referring to L_0 as the Lagrangian for a free anelastic body is that there is another contribution to the total Lagrangian that is associated with the elastic properties. This contribution arises because we have to provide for the work done on boundaries by imposed boundary tractions. A proper accounting for such effects is somewhat subtle and requires choice of a null-class Lagrangian (degenerate Lagrangian) in order to leave the field equations (Euler-Lagrange equations) unchanged while changing the natural Neumann data for the problem. The reader is therefore referred directly to the literature[9, 10, 8 (Section 3.18)], and the results that obtain after minimal replacement will simply be written down. Let $\overline{S}{}^A_i$ be the components of the Piola-Kirchhoff stress and $\overline{P}_i$ be the components of linear momentum that are associated with the elastic body with Lagrangian l_0; that is,

$$\overline{S}{}^A_i = -\partial l_0/\partial(\partial_A\overline{\chi}^i), \qquad \overline{P}_i = \partial l_0/\partial(\partial_4\overline{\chi}^i). \tag{3.5}$$

The quantities $\{\overline{\chi}^i(X^a)\}$ are then determined by solving the equations of balance of linear momentum,

$$\partial_4\overline{P}_i = \partial_A\overline{S}{}^A_i, \tag{3.6}$$

subject to the given initial and boundary data (both geometric and traction) that define the specific problem of interest. When these functions $\{\overline{\chi}^i(X^a)\}$ are put back into (3.5), we obtain the specific functions $\{P_i(X^a)\}$ and $\{S^A_i(X^a)\}$ that quantify the fields of linear momentum and Piola-Kirchhoff stress for the specific problem of interest.

This construction has two purposes. First, it serves to define the boundary traction 2-forms

$$T_i(X^a)\,|_{\partial B} = (S^A_i(X^b)\mu_A)\,|_{\partial B} \tag{3.7}$$

for the boundary traction problem that is equivalent to the specific problem of interest. Second, it serves to define the "traction Lagrangian," L_{TR}, that is the Lagrangian that accounts for the work done by the equivalent tractions (3.7) for the given problem. The explicit expression is

$$L_{TR} = B^i_A\, S^A_i - B^i_4\, P_i. \tag{3.8}$$

The last part of the Lagrangian arises through a minimal coupling argument[8] for the compensating fields $\{\phi^i_a\}$. The specific evaluation is

$$L_\phi = -\frac{s}{2}\,\delta_{ij}\, D^i_{ab}\, k^{ac}\, k^{bd}\, D^j_{cd}. \tag{3.9}$$

where s is a coupling constant, $((k^{ab})) = \mathrm{diag}(-1, -1, -1, 1/y)$, and y is a further material constant. The total Lagrangian for dislocated materials is thus of the form

$$L = L_0 + L_{TR} + L_\phi. \tag{3.10}$$

The field equations of defect dynamics are simply the Euler-Lagrange equations that obtain from the Lagrangian L in the field variables $\{\chi^i, \phi^i_a\}$. The field equations associated with the $\{\chi^i\}$ variables are

$$\partial_4 p_i = \partial_A \sigma_i^A, \tag{3.11}$$

where

$$p_i = \partial L_0 / \partial B_4^i \tag{3.12}$$

is the total linear momentum and

$$\sigma_i^A = -\partial L_0 / \partial B_A^i \tag{3.13}$$

is the total Piola-Kirchhoff stress. The boundary and initial conditions for these equations are

$$(\sigma_i^A \mu_A)\,|_{\partial B} = T_i(X^a)\,|_{\partial B}, \tag{3.14}$$

where the $\{T_i(X^a)\}$ are given by (3.7), and

$$p_i\,|_{T=T_0} = P_i\,|_{T=T_0}. \tag{3.15}$$

Thus, the anelastic fields $\{\sigma_i^A, p_i\}$ and the elastic fields $\{\bar{S}_i^A, \bar{P}_i\}$ satisfy the same initial and boundary data that is generated by $\{S_i^A(X^b), P_i(X^b)\}$ in the manner described above.

The field equations associated with the $\{\phi_a^i\}$ fields are

$$2s\,\delta_{ij}\delta^{BD}\{\partial_A(\partial_A\phi_D^j - \partial_D\phi_A^j) - \frac{1}{y}\partial_4(\partial_4\phi_D^j - \partial_D\phi_4^j)\} = \sigma_i^B - S_i^B, \tag{3.16}$$

$$\frac{2s}{y}\delta_{ij}\partial_A(\partial_A\phi_4^j - \partial_4\phi_A^j) = p_i - P_i, \tag{3.17}$$

subject to the boundary data

$$\begin{aligned} \delta^{BD}\{(\partial_A\phi_D^i - \partial_D\phi_A^i)\mu_B\}\,|_{\partial B} &= 0, \\ \delta^{BD}\{(\partial_4\phi_D^i - \partial_D\phi_4^i)\mu_B\}\,|_{\partial B} &= 0, \end{aligned} \tag{3.18}$$

and the initial data

$$(\partial_4\phi_D^i - \partial_D\phi_4^i)\,|_{T=T_0} = 0. \tag{3.19}$$

The reader should carefully note that (3.16), (3.17) show that the dislocation fields are driven by the *effective stress* $\sigma_i^B - S_i^B$ and *effective momentum* $p_i - P_i$, rather than by the total stress σ_i^B and total momentum p_i. Further, integration of (3.16) and (3.17) over the body and use of Stokes' theorem and the boundary conditions (3.18) show that all components of the effective stress and all components of the effective momentum have zero mean values on the body. Finally, since $\{S_i^A, P_i\}$ satisfy the corresponding initial-boundary value problem of elasticity, (3.16) and (3.17) give

$$\sigma_i^B = S_i^B + {}_a\sigma_i^B, \qquad p_i = P_i + {}_a p_i, \tag{3.20}$$

where $\{{}_a\sigma_i^B\}$ is the anelastic or plastic stress and $\{{}_a p_i\}$ is the anelastic or plastic momentum. It is clear from the way in which this decomposition has been effected that it is *unique* and that we have the specific evaluations

$${}_a\sigma_i^B = 2s\,\delta_{ij}\delta^{BD}\left\{\partial_A(\partial_A\phi_D^j - \partial_D\phi_A^j) - \frac{1}{y}\partial_4(\partial_4\phi_D^j - \partial_D\phi_4^j)\right\}, \tag{3.21}$$

$$ {}_a p_i = \frac{2s}{y} \delta_{ij} \partial_A (\partial_A \phi_4^j - \partial_4 \phi_A^j), \tag{3.22} $$

where $\{\phi_a^i(X^b)\}$ are solutions of the eqns (3.16)–(3.19).

An examination of the eqns (3.11)–(3.22) shows that they constitute a well-posted initial-boundary value problem for the determination of the fields $\{\chi^i(X^a), \phi_a^i(X^a)\}$ provided four independent gauge conditions are imposed on the functions $\phi_a^i(X^a)$. Accordingly, any mixed initial-boundary value problem for a material body becomes a well-posed problem for an anelastic or dislocated body, provided the actual force and displacement boundary conditions are converted to equivalent traction boundary conditions in the manner described above. Reflection shows that this is a reasonable state of affairs. There are only equivalent traction boundary conditions for anelastic bodies. In fact, it is problematic to say just what a displacement boundary condition would mean for a dislocated body because the distortion 1-forms for such a body are non-integrable and thus do not obtain as differentials of mappings from the reference configuration to the current configuration. From this point of view, the fields $\{\chi^i(X^b), \phi_a^i(X^b)\}$ are simply a system of state variables that allow us to calculate the fields of total stress and total momentum of a body in terms of the initial and boundary data. Thus, since we use Lagrangian coordinates $\{X^1, X^2, X^3\}$, the state of stress and momentum at any particle can be calculated without having to address the vexing question of just where that particle is in the dislocated body at the present time. Granted, any particle is somewhere at the present time, but we don't really have to know just where, if we can calculate the fields of stress and momentum it sees at any given time.

4. VARIATIONAL FOUNDATION FOR CALCULATION OF INTERACTION FORCES

One of the marked advantages of the gauge theory of dislocations is the ease with which we can calculate the various forces of interaction. The reason for this is the variational principle that underlies the gauge theory. Results of this nature are of sufficient importance that they warrant an independent and general derivation.

Consider a physical system whose state is described by a system of fields $\{q^\alpha(X^a) \mid 1 \le \alpha \le N\}$. The Einstein summation convention is assumed to hold with respect to both Latin and Greek indices. We assume that the dynamics of states is governed by a homogeneous variational principle with Lagrangian function $L(X^a, q^\alpha(X^a), \partial_b q^\alpha(X^a))$, and hence the q^α's satisfy the corresponding system of Euler-Lagrange equations

$$ \frac{\partial L}{\partial q^\alpha} - \frac{\partial}{\partial X^b}\left\{\frac{\partial L}{\partial(\partial_b q^\alpha)}\right\} = 0. \tag{4.1} $$

The components of the momentum-energy complex for our dynamical system are defined by

$$ T_b^a(L) = \frac{\partial L}{\partial(\partial_a q^\alpha)} \partial_b q^\alpha - \delta_b^a L. \tag{4.2} $$

Let $\partial_b^* f$ denote the explicit partial derivative of $f(X^a, q^\alpha(X^a), \partial_b q^\alpha(X^a))$ with respect to X^b; that is, the partial derivative of f with respect to X^b for constant $q^\alpha(X^a)$ and $\partial_b q^\alpha(X^a)$. A straightforward calculation based on (4.1) and (4.2) shows that

$$ \frac{\partial}{\partial X^a}(T_b^a(L)) + \partial_b^* L = 0 \tag{4.3} $$

are satisfied identically whenever the state variables satisfy the Euler-Lagrange equations (4.1) and have continuous second derivatives.

Next, we note that $\{T^a_b(L)\}$ is linear in L,

$$T^a_b(L_1 + L_2) = T^a_b(L_1) + T^a_b(L_2). \tag{4.4}$$

Thus, if the total Lagrangian function L for a system is the sum of two Lagrangian functions L_1 and L_2,

$$L = L_1 + L_2, \tag{4.5}$$

we have

$$T^a_b(L) = T^a_b(L_1) + T^a_b(L_2). \tag{4.6}$$

Suppose that L_i and L_2 are Lagrangian functions that characterize two different aspects of our physical system; for example, L_1 describes the elastic aspects of a body, while L_2 describes the dislocation aspects. In this case, $\{T^a_b(L)\}$ will satisfy (4.3), and hence (4.6) gives

$$\frac{\partial}{\partial X^a}(T^a_b(L_1)) + \partial^*_b L_1 + \frac{\partial}{\partial X^a}(T^a_b(L_2)) + \partial^*_b L_2 = 0. \tag{4.7}$$

Thus, if we set

$$F_b(L_1) = \frac{\partial}{\partial X^a}(T^a_b(L_1)) + \partial^*_b L_1, \tag{4.8}$$

$$F_b(L_2) = \frac{\partial}{\partial X^a}(T^a_b(L_2)) + \partial^*_b L_2, \tag{4.9}$$

the relations (4.7) go over into

$$F_b(L_1) + F_b(L_2) = 0. \tag{4.10}$$

Only the sum $F_b(L_1) + F_b(L_2)$ vanishes in general because the total Lagrangian function for the system is $L_1 + L_2$, not L_1 alone nor L_2 alone. Thus, $F_b(L_1)$ does not necessarily vanish by itself, and likewise with $F_b(L_2)$; they are at most self-equilibrating, $F_b(L_1) = -F_b(L_2)$.

It is clear on physical grounds that the nonvanishing of $F_b(L_1)$ and $F_b(L_2)$ arise because of interactions between the two aspects of the body that are described by L_1 and L_2. In fact, we will now prove that $F_b(L_1)$ are the components of the force (a = 1, 2, 3) and rate of work (a = 4) that are applied to the first aspect of the body as a consequence of interactions with the second aspect.

The Lagrangian function for the complete system is given by $L = L_1 + L_2$. When this is substituted into the Euler-Lagrange equations (4.1) and we define the quantities Q_α by

$$Q_\alpha = \frac{\partial}{\partial X^a}\left(\frac{\partial L_2}{\partial(\partial_\alpha q^\alpha)}\right) - \frac{\partial L_2}{\partial q^\alpha}, \tag{4.11}$$

we obtain

$$\frac{\partial L_1}{\partial q^\alpha} - \frac{\partial}{\partial X^a}\left(\frac{\partial L_1}{\partial(\partial_a q^\alpha)}\right) = Q_\alpha. \tag{4.12}$$

We may thus interpret the quantities $\{Q_\alpha\}$ as a system of generalized forces that act on the system with Lagrangian function L_1 due to the presence of the second aspect of the body with Lagrangian function L_2. Starting with (4.12) and $\{T_b^a(L_1)\}$, it is easily seen that we now have

$$\frac{\partial}{\partial X^a}(T_b^a(L_1)) + \partial_b^* L_1 = Q_\alpha \frac{\partial q^\alpha}{\partial X^b} \tag{4.13}$$

in place of (4.3). Thus, since $\{Q_\alpha\}$ is a system of generalized forces,

$$F_b(L_1) = Q_\alpha \frac{\partial q^\alpha}{\partial X^b} \tag{4.14}$$

are the physical components of the force ($b = 1, 2, 3$) and rate of work ($b = 4$) that act on the aspect of the body with Lagrangian function L_1 as a consequence of the aspect of the body with Lagrangian function L_2. *The quantities* $\{F_b(L_1)\}$ *and* $\{F_b(L_2)\}$ *are thus shown to be the physical components of the interaction forces and their rates of work.* Further, these interaction forces and energy supplies have been shown to be self-equilibrating (i.e. $F_b(L_1) + F_b(L_2) = 0$). They thus satisfy the principle of equal action and reaction for the forces ($b = 1, 2, 3$), while $F_4(L_1) + F_4(L_2) = 0$ shows that the total system is energetically closed. On the other hand, if $F_4(L_1) \neq 0$, the first aspect of the body will lose or gain energy from the second aspect. This last point is practicularly important, for one of the mistaken "folk lore" statements about systems with a variational principle is that any aspect of the resulting system is conservative. Granted, the total system is energetically closed, but not each aspect of a physical system. Thus, a total system with both elastic and dislocation modes of response will be energetically closed if it is described by a homogeneous variational principle (as is the case with the gauge theory of defects), but the elastic modes of response can and will lose energy to the dislocation modes of response if $F_4(L_0 + L_{TR}) \neq 0$.

There is one further point that needs to be made at this juncture. The relations (4.8) and (4.9) show that the interaction forces and rate of work for any Lagrangian function $\bar{L}$ have the form

$$F_b(\bar{L}) = \frac{\partial}{\partial X^a}(T_b^a(\bar{L})) + \partial_b^* \bar{L} \tag{4.15}$$

where

$$T_b^a(\bar{L}) = \frac{\partial \bar{L}}{\partial(\partial_a q^\alpha)} \partial_b q^\alpha - \delta_b^a \bar{L}. \tag{4.16}$$

It is then a simple matter to see that (4.16) will always lead to an explicit evaluation of the form

$$\frac{\partial}{\partial X^a}(T_b^a(\bar{L})) = f_b(\bar{L}) - \partial_b^* \bar{L}. \tag{4.17}$$

This allows us to interpret

$$\frac{\partial}{\partial X^a}(T_b^a(\bar{L}))$$

as the *total* forces and *total* rate of work that act on the aspect of the field that is described by the Lagrangian function $\bar{L}$. This total force and total rate of work are each made up of two parts, in view of (4.17). The first part, $f_b(\bar{L})$, is that part that comes from the presence of other aspects of the body, while the second part, $-\partial_b^* \bar{L}$, is the force and rate of work that acts on the $\bar{L}$-aspect because of spatial and temporial inhomogeneities of the $\bar{L}$-aspect. It then follows from (4.15) that $\partial_b^* \bar{L}$ may be interpreted as the forces that act on the spatial and temporal inhomogeneities of the $\bar{L}$-aspect of the body because of the presence of the fields. This interpretation is substantiated by combining (4.15) and (4.17), in which case we have

$$F_b(\overline{L}) = f_b(\overline{L}) - \partial_b^*\overline{L} + \partial_b^*\overline{L} = f_b(\overline{L}). \tag{4.18}$$

Thus, although we can explicitly calculate the forces on the inhomogeneities and the forces that the inhomogeneities apply to the fields, these inhomogeneity forces are self-equilibrating and only the part $f_b(\overline{L})$ is left in the expression for the total interaction forces. The forces associated with the inhomogeneities thus have no effect in the final evaluation of the interaction forces and rate of work between different aspects of a material body.

The simplest way of appreciating the intrinsic differences between $F_b(\overline{L})$ and $\partial_b^*\overline{L}$ is through an explicit example. Consider a system with one degree of freedom $q(T)$ and only one independent variable $T = X^4$. If the kinetic and generalized potential energies have the forms

$$\mathcal{T} = \frac{1}{2}\, m(T) \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)^2, \qquad V = -\int_a^q f(T, \lambda)\, \mathrm{d}\lambda, \tag{4.19}$$

so that $L = \mathcal{T} - V$, then the Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{m(T)\frac{\mathrm{d}q}{\mathrm{d}T}\right\} = f(t, q). \tag{4.20}$$

If this equation is multiplied by $\mathrm{d}q/\mathrm{d}T$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}T}(\mathcal{T} + V) &= -\frac{1}{2}\frac{\mathrm{d}m}{\mathrm{d}T}\left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)^2 - \int_a^q \frac{\partial}{\partial T} f(T, \lambda)\, \mathrm{d}\lambda \\ &= -\,\partial_T^*(\mathcal{T} - V) = -\,\partial_T^*\overline{L}.\end{aligned} \tag{4.21}$$

On the other hand,

$$T_b^a(\overline{L}) = \delta_4^a\, \delta_b^4\, (\mathcal{T} + V) \tag{4.22}$$

and hence (4.21) and (4.22) give

$$\frac{\partial}{\partial X^a}(T_b^a(\overline{L})) = \frac{\mathrm{d}}{\mathrm{d}T}(\mathcal{T} + V) = -\,\partial_T^*\overline{L}. \tag{4.23}$$

We now substitute (4.21) into (4.15) with $b = 4$. This gives

$$F_4(\overline{L}) = -\,\partial_T^*\overline{L} + \partial_T\overline{L} = 0 \tag{4.24}$$

as it must, because $\overline{L}$ is the total Lagrangian for this system, so that there can be no nonzero forces of interaction. Although this may seem like "robbing Peter to pay Paul," it is not, for we have secured some important new information; namely, $\partial_T\overline{L}$ *is the energy supplied by the motion to the agencies that maintain the temporal inhomogeneities* $m(T)$ *and* $f(T, q)$, *and that* $\partial_T^*\overline{L}$ *is the energy lost by the motion because of the temporal inhomogeneities.*

5. FORCES DUE TO INHOMOGENEITIES

The total Lagrangian function for a dislocated material body is given by (3.10). It has two distinct parts. The first part,

$$L_1 = L_{e1} = L_0 + L_{TR} \tag{5.1}$$

characterizes the elastic aspects of the body, while

$$L_2 = L_\Phi \tag{5.2}$$

characterizes the dislocation aspects. We may, therefore, apply the results of the last section directly to dislocation dynamics with $L = L_1 + L_2$.

The first task is that of calculating the forces and rate of work that arise from the presence of inhomogeneities; that is

$$\overset{*}{\partial}_b L_{el} = \overset{*}{\partial}_b (L_0 + L_{Tr}). \tag{5.3}$$

An inspection of (3.3) shows that the only inhomogeneities in L_0 arise through a dependence of the mass density and the elastic moduli on position and time, $\rho_0 = \rho_0(X^a)$, $\lambda = \lambda(X^a)$, $\mu = \mu(X^a)$. On the other hand, the fields $S_i^A = S_i^A(X^a)$, $P_i = P_i(X^a)$ are specific functions of the independent variables that are given by the solution to the equivalent traction, initial-value problem of elasticity. Thus, (3.8) gives

$$L_{TR} = B_A^i\, S_i^A(X^a) - B_4^i\, P_i(X^a). \tag{5.4}$$

With these observations at hand, we only need to substitute (3.3) and (5.4) into (5.3). This gives us the evaluation

$$\begin{aligned}\overset{*}{\partial}_b L_{el} &= \frac{1}{2}\frac{\partial \rho_0}{\partial X^b} B_4^i\, \delta_{ij}\, B_4^j \\ &\quad - \frac{1}{8}\left\{\frac{\partial \lambda}{\partial X^b}(E_{AB}\delta^{AB})^2 + 2\frac{\partial \mu}{\partial X^b} E_{AB}\delta^{AC}\delta^{BF}E_{CF}\right\} \\ &\quad + B_A^i \frac{\partial S_i^A}{\partial X^b} - B_4^i \frac{\partial P_i}{\partial X^b}\end{aligned} \tag{5.5}$$

of the forces that act on the inhomogeneities as a consequence of the fields $\{B_a^i\}$. The first term, $\frac{1}{2}\, \partial\rho_0/\partial X^b\, B_4^i\, \delta_{ij}\, B_4^j$, represents the forces and rate of work applied to the mass inhomogeneity. The second set of terms, in the curly brackets, are the forces and rate of work applied to inhomogeneities in the elastic compliances. The last set of terms, $B_A^i\, \partial S_i^A/\partial X^b - B_4^i\, \partial P_i/\partial X^b$, represent the forces and rate of work that come about through responses of the body to the imposed boundary tractions ($B_A^i\, \partial S_i^A/\partial X^b$) and the initial data ($B_4^i\, \partial P_i/\partial X^b$). These latter forces are not unexpected, for we have seen that a dislocated body always sees the elastic field generated by the initial and traction boundary data as an externally imposed field. What is different about these forces is that they are always present whenever $\{S_i^A, P_i\}$ are other than constant fields (free unloaded bodies in equilibrium), while the other terms in (5.5) disappear whenever the body under study has constant mass density and elastic compliances. This serves to point up the often overlooked fact that a dislocated body is aware of the imposed initial and traction boundary conditions at every point in the body and at every time.

6. INTERACTION FORCES

We noted in the last section that a dislocated body has only two essential aspects and that these two aspects are characterized by the Lagrangian function for the elastic properties, L_{el}, and the Lagrangian function for the dislocations, L_Φ,

$$L = L_{el} + L_\Phi. \tag{6.1}$$

Accordingly, we need only compute the interaction forces $F_b(L_{el})$ because the interaction forces are self-equilibrating,

$$F_b(L_\phi) = - F_b(L_{e1}). \tag{6.2}$$

The first thing to note is that $L_{e1} = L_0 + L_{TR}$. Thus, a direct calculation based on (3.3), (3.8) and (4.2) leads to an explicit evaluation of the momentum-energy complex $T_b^a(L_{e1})$. A direct calculation and use of the equations of motion (3.6), (3.16), and (3.17) then give

$$\frac{\partial}{\partial X^a}(T_b^a(L_{e1})) = (\sigma_i^A - S_i^A)\,\partial_b\phi_A^i - (p_i - P_i)\,\partial_b\phi_4^i - \partial_b^*(L_{e1}), \tag{6.3}$$

where $\partial_b^*(L_{e1})$ is given by (5.5). We then simply substitute (6.3) into (4.15) in order to obtain the required results:

$$F_b(L_{e1}) = (\sigma_i^A - S_i^A)\,\partial_b\phi_A^i - (p_i - P_i)\,\partial_b\phi_4^i. \tag{6.4}$$

The components of the forces that act on the elastic response mode as a consequence of the dislocation mode ($b = B$, $B = 1, 2, 3$) are given by

$$F_B(L_{e1}) = (\sigma_i^A - S_i^A)\,\partial_B\phi_A^i - (p_i - P_i)\,\partial_B\phi_4^i. \tag{6.5}$$

The interaction forces are thus governed by the effective stress, $\sigma_i^A - S_i^A$, and the effective linear momentum, $p_i - P_i$, rather than by the total stress and total linear momentum, (σ_i^A, p_i). However, for static bodies of infinite spatial extent that are traction-free at infinity (the classical case, $S_i^A = 0$, $P_i = 0$) the effective stress and linear momentum agree with the total stress and linear momentum. In this case, it has been previously shown that (6.5) reproduce the negative of Peach-Koeler forces (the reaction to the Peach-Koeler force). [5, 6, 8]. It thus follows directly from (6.5) that *the Peach-Koeler forces can be computed for finite bodies with applied tractions by the simple expedient of replacing total stress by effective stress.*

The rate of work performed by the interaction forces is obtained from (6.4) by setting $b = 4$:

$$F_4(L_{e1}) = (\sigma_i^A - S_i^A)\,\partial_4\phi_A^i - (p_i - P_i)\,\partial_4\phi_4^i. \tag{6.6}$$

This is most easily interpreted by noting that

$$B_A^i = \partial_A\chi^i + \phi_A^i, \qquad V^i = \partial_4\chi^i + \phi_4^i$$

and then setting $\chi^i = \delta_A^i X^A + u^i(X^b)$ where $\{u_i(X^b)\}$ are the displacement functions. It is then natural, and indeed customary, to define the plastic distortion, P/B_A^i, and plastic velocity, P/V^i, by

$$\partial_A\chi^i = B_A^i + P/B_A^i, \qquad \partial_4\chi^i = V^i + P/V^i.$$

This leads to the explicit evaluations

$$P/B_A^i = -\phi_A^i, \qquad P/V^i = -\phi_4^i. \tag{6.7}$$

Further, if $B^i = B_a^i\,dX^a$ is divided into exact and anti-exact parts, as discussed in [8], then the decomposition leading to (6.7) is unique for a given choice of center for the linear homotopy operator. It is now a simple matter to combine (6.6) and (6.7) in order to obtain

$$F_4(L_{el}) = -(\sigma_i^A - S_i^A)\,\partial_4(P/B_A^i) + (p_i - P_i)\,\partial_4(P/V^i). \tag{6.8}$$

This shows that *the elastic response loses energy to the dislocation fields by exactly the rate at which the effective stress performs plastic work* in a steady plastic flow ($\partial_4(P/V^i) = 0$). The principle ingredient of Drucker's postulate is thus an immediate consequence of the theory, provided we replace total stress by effective stress and total linear momentum by effective linear momentum. Particular note should be made here that (6.8) is a consequence of the full dynamics of the gauge theory of dislocated bodies, and thus constitutes a direct extension to the transient, unsteady situation.

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